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Galilei covariance and quasi-free representations of Fermi fields

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Abstract. Some quasi-free irreducible representations of the CAR algebra of a non-relativistic free Fermi field are constructed, which are covariant under certain subgroups of the extended Galilei group. The construction of fully covariant (but reducible) representations as direct integrals of partially covariant ones is described. This method is also applicable to arbitrary C^*-algebras and other covariance groups.

1. Introduction

Covariance properties of quasi-free representations of Fermi fields have been investigated recently (Kraus and Streater 1981). The general theory was illustrated there by the particular example of relativistic fermions, with the inhomogeneous Lorentz group as covariance group. The non-relativistic case (i.e. Galilei covariance) has been investigated by Stark (1981). Some partially covariant models, as constructed there, are discussed here in § 2, whereas § 3 describes a generalised version of the direct integral construction used by Stark (1981) to obtain fully covariant models from partially covariant ones. This method is also applicable to bosons.

We first recall some basic facts. (For more details and references to the original literature see Kraus and Streater (1981).) The CAR algebra $\mathfrak{A}$—the field algebra of a Fermi field—is generated by the annihilation and creation operators $a(\varphi)$ and $a(\varphi)^*$, which satisfy the canonical anticommutation relations

$$a(\varphi)a(\psi)+a(\psi)a(\varphi)=0, \qquad a(\varphi)a(\psi)^*+a(\psi)^*a(\varphi)=(\varphi,\psi). \tag{1}$$

Here φ and ψ are one-particle state vectors from a separable Hilbert space $\mathscr{K}$, $a(\varphi)$ depends antilinearly (and thus $a(\varphi)^*$ linearly) on φ, and (φ, ψ) denotes the scalar product in $\mathscr{K}$.

Given a state (i.e. a positive linear functional) ω on $\mathfrak{A}$, we obtain from it a cyclic representation π_ω of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}_\omega$ by the well known GNS construction (see e.g. Emch (1972)). We study here the so-called quasi-free states ω_A, defined by

$$\omega_A(a(\varphi_n)^* \ldots a(\varphi_1)^* a(\psi_1) \ldots a(\psi_m)) = \delta_{nm} \det[(\psi_i, A\varphi_k)] \tag{2}$$

in terms of an operator A on $\mathscr{H}$ with $0 \leqslant A \leqslant 1$. A quasi-free state ω_A is pure, thus leading to an irreducible representation $\pi_{\omega_A} \equiv \pi_A$ on $\mathscr{H}_{\omega_A} \equiv \mathscr{H}_A$, if and only if A is a projection operator. We consider this case only. Two such representations π_A and π_B are unitarily equivalent if and only if $A-B$ belongs to $B(\mathscr{K})_2$, the Hilbert–Schmidt class of operators on $\mathscr{K}$. Physically speaking, equation (2) describes a state of the Fermi field in which all one-particle states in the subspace $A\mathscr{K}$ of $\mathscr{K}$ are occupied, whereas all states in $(1-A)\mathscr{K}$ are unoccupied. Thus, in particular, $A=0$ yields the vacuum state ω_0 and the Fock representation π_0, whereas $A=1$ describes the 'plenum' state ω_1, corresponding to the 'anti-Fock' representation π_1.

Symmetries of the one-particle space $\mathscr{K}$ induce automorphisms of the CAR algebra $\mathfrak{A}$. Namely, if V_g $(g \in \mathrm{G})$ is a continuous unitary representation of a symmetry group G on $\mathscr{K}$, then the automorphisms τ_g of $\mathfrak{A}$ defined by

$$\tau_g(a(\varphi)) = a(V_g\varphi) \tag{3}$$

represent G in Aut $\mathfrak{A}$, the group of automorphisms of $\mathfrak{A}$. With C denoting a subgroup of G, a representation π of $\mathfrak{A}$ on a representation space $\mathscr{H}$ is called C-covariant if the automorphisms $\tau_g(g \in \mathrm{C})$ are unitarily implemented on $\mathscr{H}$, i.e. if there is a locally continuous unitary projective representation (i.e. a representation up to a factor) U_g of C on $\mathscr{H}$, such that

$$\pi(\tau_g(Y)) = U_g\pi(Y)U_g^* \qquad \text{for all } Y \in \mathfrak{A} \text{ and } g \in \mathrm{C}. \tag{4}$$

As shown by Kraus and Streater (1981), an irreducible quasi-free representation π_A is C-covariant if and only if

$$X_g \equiv A - V_gAV_g^* \in B(\mathscr{K})_2 \tag{5}$$

for all $g \in \mathrm{C}$. An equivalent condition is

$$\sum_i \left(1 - \sum_k |(\varphi_i, V_g\varphi_k)|^2\right) < \infty \tag{6}$$

for all $g \in \mathrm{C}$, with $\{\varphi_i\}$ denoting an orthogonal system spanning the subspace $A\mathscr{K}$ of occupied one-particle states. To prove the equivalence of (5) and (6), consider the projection operators $A' = 1 - A$, $A_g = V_gAV_g^*$, $A'_g = 1 - A_g$ and the non-negative operators $R_g = AA'_gA$, $R'_g = A'A_gA'$. Since $X_g^2 = R_g + R'_g$, (5) is equivalent to $R_g \in B(\mathscr{K})_1$ and $R'_g \in B(\mathscr{K})_1$, the trace class of operators on $\mathscr{K}$. Evaluating the trace with a complete orthogonal system containing the system $\{\varphi_i\}$ as a subbasis in $A\mathscr{K}$, we obtain

$$\operatorname{Tr} R_g = \sum_i (\varphi_i, A'_g\varphi_i) = \sum_i (1 - (\varphi_i, V_gAV_g^*\varphi_i)) = \sum_i \left(1 - \sum_k |(\varphi_i, V_g\varphi_k)|^2\right).$$

Therefore (5) directly implies (6). *Vice versa*, (6) implies $R_g \in B(\mathscr{K})_1$ and, with g^{-1} for g, also $R_{g^{-1}} = AA'_{g^{-1}}A = (A'_{g^{-1}}A)^*(A'_{g^{-1}}A) \in B(\mathscr{K})_1$. This, being equivalent to $A'_{g^{-1}}A \in B(\mathscr{K})_2$, implies

$$V_g(A'_{g^{-1}}A)V_g^* = A'A_g \in B(\mathscr{K})_2,$$

and thus, finally, also

$$A'A_g(A'A_g)^* = R'_g \in B(\mathscr{K})_1.$$

2. Partially Galilei covariant models

The (inhomogeneous) Galilei group Γ of classical mechanics consists of the coordinate transformations

$$\gamma(\tau, \boldsymbol{a}, \boldsymbol{v}, R)\colon \boldsymbol{x} \to \boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a},\ t \to t' = t + \tau \tag{7}$$

containing a spatial rotation R, a boost with velocity $\boldsymbol{v}$, a space translation by $\boldsymbol{a}$, and a time translation by τ, and satisfying the multiplication law

$$\gamma(\tau, \boldsymbol{a}, \boldsymbol{v}, R)\gamma(\sigma, \boldsymbol{b}, \boldsymbol{w}, S) = \gamma(\tau+\sigma, \boldsymbol{a}+R\boldsymbol{b}+\sigma\boldsymbol{v}, \boldsymbol{v}+R\boldsymbol{w}, RS). \tag{8}$$

For quantum mechanics, the covering group $\bar{\Gamma}$ of Γ—also called 'Galilei group' here—is more important than Γ itself. This covering group $\bar{\Gamma}$ results from Γ by reinterpreting the symbol R occurring in a 'Galilei transformation' $\gamma(\tau, \boldsymbol{a}, \boldsymbol{v}, R) \in \bar{\Gamma}$ as an SU(2), rather than an SO(3) matrix. In order to simplify the notation, however, R will still be interpreted as the SO(3) matrix corresponding to $R \in \mathrm{SU}(2)$ via the well known homomorphism, whenever R acts on spatial vectors like $\boldsymbol{b}$ or $\boldsymbol{w}$. With this notation, the group multiplication law has the same form (8) in Γ and in $\bar{\Gamma}$.

The symmetry group G of a free particle of mass m in non-relativistic quantum mechanics (see e.g. Lévy-Leblond 1971) is not the group $\bar{\Gamma}$, but rather a central extension of it, whose elements

$$g = (\mathrm{e}^{\mathrm{i}\alpha}, \gamma) = g(\alpha, \tau, \boldsymbol{a}, \boldsymbol{v}, R)$$

consist of a Galilei transformation $\gamma = \gamma(\tau, \boldsymbol{a}, \boldsymbol{v}, R) \in \bar{\Gamma}$ and a phase factor $\mathrm{e}^{\mathrm{i}\alpha}$. The multiplication law in G reads

$$(\mathrm{e}^{\mathrm{i}\alpha}, \gamma)(\mathrm{e}^{\mathrm{i}\beta}, \gamma') = (\mathrm{e}^{\mathrm{i}(\alpha+\beta+\xi(\gamma,\gamma'))}, \gamma\gamma'), \tag{9}$$

or

$$g(\alpha, \tau, \boldsymbol{a}, \boldsymbol{v}, R)g(\beta, \sigma, \boldsymbol{b}, \boldsymbol{w}, S) = g(\alpha+\beta+\xi(\gamma, \gamma'), \tau+\sigma, \boldsymbol{a}+R\boldsymbol{b}+\sigma\boldsymbol{v}, \boldsymbol{v}+R\boldsymbol{w}, RS), \tag{10}$$

with $\gamma' = \gamma(\sigma, \boldsymbol{b}, \boldsymbol{w}, S)$ and

$$\xi(\gamma, \gamma') = \tfrac{1}{2}mv^2\sigma + m\boldsymbol{v}\cdot R\boldsymbol{b}. \tag{11}$$

We call G the extended Galilei group.

The state space $\mathscr{H}$ of a particle of spin s may be represented as the space of $(2s+1)$-component square-integrable momentum space wavefunctions $\varphi_\mu(\boldsymbol{p})$, $\mu = 1, \ldots, 2s+1$. On such wavefunctions, the unitary operator V_g representing the group element $g = g(\alpha, \tau, \boldsymbol{a}, \boldsymbol{v}, R)$ acts as (Lévy-Leblond 1971)

$$(V_g\varphi)_\mu(\boldsymbol{p}) = \exp\left[\mathrm{i}\left(\alpha + \frac{p^2}{2m}\tau - \boldsymbol{p}\cdot\boldsymbol{a}\right)\right]\sum_{\nu=1}^{2s+1} D^s_{\mu\nu}(R)\varphi_\nu(R^{-1}(\boldsymbol{p}-m\boldsymbol{v})), \tag{12}$$

with $D^s_{\mu\nu}(R)$ denoting the matrix elements of the spin-s representation D^s of SU(2). By identifying $\mathscr{H}$, in the usual way, with the tensor product $\mathscr{H}^0 \otimes \mathbb{C}^{2s+1}$ of the 'orbit' state space $\mathscr{H}^0 = L^2(\boldsymbol{p})$ of square-integrable momentum space wavefunctions $\varphi(\boldsymbol{p})$ and the $(2s+1)$-dimensional spin space $\mathbb{C}^{2s+1}$, (12) may be written as

$$V_g = V^0_g \otimes D^s(R) \tag{13}$$

with the representation V^0_g of G on $\mathscr{H}^0$ given by

$$(V^0_g\varphi)(\boldsymbol{p}) = \exp[\mathrm{i}(\alpha + p^2\tau/2m - \boldsymbol{p}\cdot\boldsymbol{a})]\varphi(R^{-1}(\boldsymbol{p}-m\boldsymbol{v})) \tag{14}$$

and corresponding to a particle of spin zero.

The projection operators A studied here are taken to be of the particular form

$$A = A^0 \otimes 1 \tag{15}$$

with a projection operator A^0 on $\mathscr{H}^0$ and the unit matrix 1 in spin space. For such

A the covariance condition (5) becomes

$$A - V_g A V_g^* = (A^0 - V_g^0 A^0 V_g^{0*}) \otimes 1 \in B(\mathscr{K})_2,$$

by virtue of (13), (15) and the unitarity of the matrices $D^s(R)$. Since the space $\mathbb{C}^{2s+1}$ is of finite dimension, (5) thus becomes equivalent to

$$X_g^0 \equiv A^0 - V_g^0 A^0 V_g^{0*} \in B(\mathscr{K}^0)_2,$$

which is nothing but the covariance condition (5) for the spin-zero representation (14). It is thus sufficient to investigate further the case of spinless particles only, because the results so obtained can be applied immediately, via (15), also to the general case of arbitrary spin. By this restriction to the case $s = 0$, the superscript 0 on $\mathscr{K}^0$, V_g^0 and A^0 becomes superfluous, and is thus omitted from now on.

The easiest way of satisfying the covariance condition (5) for some 'covariance' subgroup $\mathrm{C} \subseteq \mathrm{G}$ is to choose A invariant under C, i.e. to satisfy

$$[A, V_g] = 0 \qquad \text{for all } g \in \mathrm{C}, \tag{16}$$

such that $X_g \equiv 0$ on C. In this case, equations (2) and (3) imply

$$\omega_A(\tau_g(Y)) = \omega_A(Y) \qquad \text{for all } Y \in \mathfrak{A} \text{ and } g \in \mathrm{C}, \tag{17}$$

i.e. ω_A is a C-invariant state; *vice versa*, (17) implies (16), by (2). The C-covariance of a representation π_ω corresponding to a C-invariant state ω also follows directly by the GNS construction, and the implementing operators U_g, $g \in \mathrm{C}$ on $\mathscr{H}_\omega$ form a true (rather than only a projective) representation of C in this case (Emch 1972).

Since the representation V of G considered here is irreducible, the only G-invariant projection operators are $A = 0$ and $A = 1$, leading to the Fock and anti-Fock representations, respectively, which are already known to be covariant. Non-trivial projection operators A can thus be invariant under proper subgroups C of G only.

Every projection operator A is invariant under the central subgroup P of pure phase factors (i.e. of elements $g = (\mathrm{e}^{\mathrm{i}\alpha}, \gamma) \in \mathrm{G}$ with $\gamma = e$, the unit element of $\bar{\Gamma}$), which according to (14) act multiplicatively on $\mathscr{K}$. The corresponding automorphisms $\tau_g (g \in \mathrm{P})$ of $\mathfrak{A}$—the so-called gauge transformations—are thus implemented in every representation π_A. But again this is well known.

Less trivial—and physically more interesting—subgroups of G are the Euclidean group E and the homogeneous Galilei group H, as obtained from G by restricting the parameters of group elements $g(\alpha, \tau, \boldsymbol{a}, \boldsymbol{v}, R)$ by the conditions $\alpha = 0$, $\tau = 0$, $\boldsymbol{v} = 0$ for E, and $\alpha = 0$, $\tau = 0$, $\boldsymbol{a} = 0$ for H. We may also consider them as subgroups of the Galilei group $\bar{\Gamma}$. Since R is still an SU(2) rather than an SO(3) matrix, we are actually dealing with the covering groups of what are usually called Euclidean and homogeneous Galilei groups.

For the Euclidean group, with $g = g(0, 0, \boldsymbol{a}, 0, R)$, equation (14) yields

$$(V_g \varphi)(\boldsymbol{p}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{a}} \varphi(R^{-1}\boldsymbol{p}). \tag{18}$$

The representation of the homogeneous Galilei group H has a correspondingly simple form in the configuration space representation, i.e. in terms of the Fourier transformed wavefunctions

$$\tilde{\varphi}(\boldsymbol{x}) = (2\pi)^{-3/2} \int \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \varphi(\boldsymbol{p})\, \mathrm{d}^3\boldsymbol{p}. \tag{19}$$

For $g = g(0, 0, 0, \boldsymbol{v}, R) \in \mathrm{H}$, (14) and (19) yield

$$(V_g\tilde{\varphi})(\boldsymbol{x}) = \mathrm{e}^{im\boldsymbol{v}\cdot\boldsymbol{x}}\tilde{\varphi}(R^{-1}\boldsymbol{x}). \tag{20}$$

By (18), projection operators of the form

$$A = \chi_\Omega(\boldsymbol{p}) \tag{21}$$

are invariant under all Euclidean transformations V_g, $g \in \mathrm{E}$. Here $\chi_\Omega(\boldsymbol{p})$ is the characteristic function of an arbitrary spherically symmetric volume Ω in momentum space,

$$\chi_\Omega(\boldsymbol{p}) = \begin{cases} 1 & \boldsymbol{p} \in \Omega \\ 0 & \boldsymbol{p} \notin \Omega \end{cases} \qquad R\Omega = \Omega \text{ for all } R \in \mathrm{SO}(3),$$

and acts as multiplication operator on $\mathcal{H} = L^2(\boldsymbol{p})$. Moreover, such operators A are invariant also under the time translation subgroup T_1 of G ($\alpha = 0$, $\boldsymbol{a} = 0$, $\boldsymbol{v} = 0$, $R = 1$)—as is almost obvious from (14)—and under the 'gauge' subgroup P (see above). According to (10) and (11), the three subgroups E, T_1 and P commute with each other, thus generating a subgroup $\tilde{\mathrm{E}}$ of G isomorphic to the direct product $\mathrm{E} \otimes \mathrm{T}_1 \otimes \mathrm{P}$. Every A of the form (21) thus leads to an $\tilde{\mathrm{E}}$-covariant representation π_A, and as ω_A is invariant under $\tilde{\mathrm{E}}$, the implementing operators U_g on $\mathcal{H}_A$ form a true (rather than only a projective) representation of $\tilde{\mathrm{E}}$.

If, in addition, the subgroup $\mathrm{B} \subset \mathrm{G}$ of pure boosts ($\alpha = 0$, $\tau = 0$, $\boldsymbol{a} = 0$, $R = 1$) were also implemented, then the representations π_A would be covariant under the whole extended Galilei group G. This, however, is not the case. For a pure boost, $g = g(0, 0, 0, \boldsymbol{v}, 1)$, V_g acts according to (14) as a translation operator:

$$(V_g\varphi)(\boldsymbol{p}) = \varphi(\boldsymbol{p} - m\boldsymbol{v}).$$

With A given by (21), we therefore obtain

$$A - V_g A V_g^* = \chi_\Omega(\boldsymbol{p}) - \chi_\Omega(\boldsymbol{p} - m\boldsymbol{v}),$$

which for $\boldsymbol{v} \neq 0$ is a Hilbert–Schmidt operator only in the trivial cases where Ω is either empty (and thus $A = 0$) or the whole of momentum space (and thus $A = 1$). Therefore the covariance condition (5) is violated for boosts unless A is trivial.

The close analogy of equations (18) and (20) implies that projection operators of the form

$$A = \chi_\Omega(\boldsymbol{x}), \tag{22}$$

now acting as multiplication operators on the configuration space wavefunctions $\tilde{\varphi}(\boldsymbol{x})$, are invariant under gauge and homogeneous Galilei transformations $V_g (g \in \mathrm{P}$ or $\mathrm{H})$, and thus also under the subgroup H′ of G generated by P and H. As H and P commute, by (10), H′ is isomorphic to $\mathrm{H} \otimes \mathrm{P}$. The states ω_A obtained from (22) are thus H′-invariant, and the corresponding representations H′-covariant, with implementing operators U_g representing H′ on $\mathcal{H}_A$.

Moreover, since by (14) and (19) the space translation subgroup T_3 of G (with $\alpha = 0$, $\tau = 0$, $\boldsymbol{v} = 0$, $R = 1$) is represented by translations $\tilde{\varphi}(\boldsymbol{x}) \to \tilde{\varphi}(\boldsymbol{x} - \boldsymbol{a})$ of the configuration space wavefunctions, it follows as above that space translations are not implemented in representations π_A with (22) unless A is either 0 or 1. The last result now implies, however, that time translations are also not implemented. To show this, assume the contrary. With (10) and (11) one easily verifies the identity

$$g(\tfrac{1}{2}mv^2\tau, 0, 0, -\boldsymbol{v}, 1)g(0, -\tau, 0, 0, 1)g(0, 0, 0, \boldsymbol{v}, 1)g(0, \tau, 0, 0, 1) = g(0, 0, \tau\boldsymbol{v}, 0, 1). \tag{23}$$

By the previous results and our assumption, the four transformations on the left-hand side of (23) are implemented in π_A. The same, then, must be true for their product, which is a pure space translation. This is impossible, however, as shown above.

In the models considered so far, the covariance condition (5) was satisfied in the stronger form $X_g \equiv 0$; i.e. the state ω_A was invariant under some 'covariance' subgroup C of G. Another class of partially Galilei covariant models can be constructed (Stark 1981) by exploiting the covariance condition (6) or rather, again, a somewhat stronger condition, namely

$$\sum_i (1 - |(\varphi_i, V_g\varphi_i)|^2) < \infty. \tag{24}$$

Since

$$0 \leq 1 - \sum_k |(\varphi_i, V_g\varphi_k)|^2 \leq 1 - |(\varphi_i, V_g\varphi_i)|^2$$

for all i, (24) indeed implies (6), and is thus, if valid for all g in some subgroup $\mathrm{C} \subseteq \mathrm{G}$, sufficient for C-covariance. The models in question are constructed by specifying the orthonormal vectors φ_i $(i = 1, 2, \ldots)$ occurring in (24). Remember that these vectors span the subspace $A\mathcal{H}$ of $\mathcal{H}$, i.e. in Dirac's notation,

$$A = \sum_i |\varphi_i\rangle\langle\varphi_i|. \tag{25}$$

An Euclidean covariant model is obtained by choosing a suitable increasing sequence of radii r_i, $i = 1, 2, \ldots$, and taking φ_i to be state vectors with the configuration space wavefunctions (cf equation (19))

$$\tilde{\varphi}_i(\boldsymbol{x}) = \begin{cases} C_i & \text{if } r_{i-1} \leq |\boldsymbol{x}| \leq r_i, \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

which are normalised if

$$|C_i|^2 = [\tfrac{4}{3}\pi(r_i^3 - r_{i-1}^3)]^{-1}. \tag{27}$$

Their mutual orthogonality is obvious. Every vector φ_i—and therefore A, as given by (25)—is invariant under rotations, $\tilde{\varphi}_i(\boldsymbol{x}) \to \tilde{\varphi}_i(R^{-1}\boldsymbol{x})$. To prove covariance under the Euclidean group E, it therefore suffices to verify condition (24) for space translations, $g = g(0, 0, \boldsymbol{a}, 0, 1) \in \mathrm{T}_3$, with $(V_g\tilde{\varphi}_i)(\boldsymbol{x}) = \tilde{\varphi}_i(\boldsymbol{x} - \boldsymbol{a})$.

If (as assumed here, see below) the differences $r_i - r_{i-1}$ increase with i, then for every fixed $\boldsymbol{a}$ there is an index i_0 such that $|\boldsymbol{a}| \equiv a < r_i - r_{i-1}$ for all $i \geq i_0$. The finitely many summands with $i < i_0$ are immaterial in (24). For $i \geq i_0$, an elementary but lengthy calculation (Stark 1981) leads to

$$\begin{aligned} 1 - |(\varphi_i, V_g\varphi_i)|^2 \\ = [\tfrac{3}{2}a(r_i^2 + r_{i-1}^2)(r_i^3 - r_{i-1}^3) - \tfrac{9}{16}a^2(r_i^2 + r_{i-1}^2)^2 \\ - \tfrac{1}{4}a^3(r_i^3 - r_{i-1}^3) + \tfrac{3}{16}a^4(r_i^2 + r_{i-1}^2) - \tfrac{1}{64}a^6](r_i^3 - r_{i-1}^3)^{-2}. \end{aligned}$$

Now choosing

$$r_i = \rho\kappa^{i^2} \tag{28}$$

with constants $\rho > 0$ and $\kappa > 1$, the quotient

$$Q_i = \frac{1-|(\varphi_{i+1}, V_g\varphi_{i+1})|^2}{1-|(\varphi_i, V_g\varphi_i)|^2}$$

becomes a rational function of κ, with a numerator and a denominator polynomial of degree $11i^2+10i+5$ and $11i^2+12i+6$, respectively. Thus $\kappa > 1$ implies $Q_i \to_i 0$, so that, according to the ratio test, the series (24) is indeed convergent.

For pure boosts, $g = g(0, 0, 0, \boldsymbol{v}, 1) \in B$, on the other hand, with

$$(V_g\tilde{\varphi}_i)(\boldsymbol{x}) = \mathrm{e}^{im\boldsymbol{v}\cdot\boldsymbol{x}}\tilde{\varphi}_i(\boldsymbol{x}), \tag{29}$$

the covariance condition (6) is violated. Namely, (26) and (29) imply $(\varphi_i, V_g\varphi_k) = 0$ for $i \neq k$, and

$$(\varphi_i, V_g\varphi_i) = |C_i|^2 \int_{r_{i-1}\le|\boldsymbol{x}|\le r_i} \mathrm{e}^{im\boldsymbol{v}\cdot\boldsymbol{x}}\,\mathrm{d}^3\boldsymbol{x}.$$

This is easily evaluated and yields, together with (27) and (28), an expression for $(\varphi_i, V_g\varphi_i)$ which converges to zero for $i \to \infty$ (Stark 1981). Therefore the series

$$\sum_i \left(1 - \sum_k |(\varphi_i, V_g\varphi_k)|^2\right) \equiv \sum_i (1 - |(\varphi_i, V_g\varphi_i)|^2)$$

is divergent.

We were unable to decide by comparably simple estimates—and therefore leave it open here—whether or not the time translation subgroup T_1 is implemented. The covariance group C of the representation π_A constructed above is thus either the group $\tilde{\mathrm{E}} \cong \mathrm{E}\otimes\mathrm{T}_1\otimes\mathrm{P}$, as for the previous model (21), or the smaller group $\mathrm{E}' \cong \mathrm{E}\otimes\mathrm{P}$.

In the latter case the implementing operators U_g, $g \in \mathrm{E}'$, on $\mathscr{H}_A$ can be chosen to form a true representation of E′. Namely, as shown by Bargmann (1954), the phase factors of the operators U_g implementing the subgroups E and P of E′ may be chosen such that these operators form true representations U_g, $g \in$ E or P, of these subgroups. Moreover, since ω_A is invariant under gauge transformations and rotations, we also know that the gauge operators $U_\alpha \equiv U_g$, $g = g(\alpha, 0, 0, 0, 1) \in \mathrm{P}$, commute with the rotation operators $U_R \equiv U_g$, $g = g(0, 0, 0, 0, R)$, since the GNS construction yields commuting gauge and rotation operators, and the U_g are unique up to phase factors. Thus there only remains to prove the relation $[U_\alpha, U_{\boldsymbol{a}}] = 0$ for arbitrary gauge operators U_α and translation operators $U_{\boldsymbol{a}} \equiv U_g$, $g = g(0, 0, \boldsymbol{a}, 0, 1) \in \mathrm{T}_3$. Since (due to the irreducibility of π_A) U_g, $g \in \mathrm{E}'$, is a projective representation of E′, we already know that

$$U_\alpha U_{\boldsymbol{a}} = \mathrm{e}^{\mathrm{i}\xi(\alpha, \boldsymbol{a})} U_{\boldsymbol{a}} U_\alpha. \tag{30}$$

From this and the representation property

$$U_{\boldsymbol{a}_1} U_{\boldsymbol{a}_2} = U_{\boldsymbol{a}_1+\boldsymbol{a}_2}$$

we easily obtain the relation

$$\xi(\alpha, \boldsymbol{a}_1) + \xi(\alpha, \boldsymbol{a}_2) = \xi(\alpha, \boldsymbol{a}_1 + \boldsymbol{a}_2),$$

which (together with continuity) implies

$$\xi(\alpha, \boldsymbol{a}) = \boldsymbol{k}(\alpha)\cdot\boldsymbol{a} \tag{31}$$

with some 'vector' $\boldsymbol{k}(\alpha)$. The operators $U_{\boldsymbol{a}}$ and U_R generate a representation of the Euclidean group E, which implies

$$U_R^{-1} U_{\boldsymbol{a}} U_R = U_{R^{-1}\boldsymbol{a}}.$$

From this, together with (30), (31) and the commutativity $[U_R, U_\alpha]=0$ (see above), we get

$$U_R^{-1} U_\alpha U_{\boldsymbol{a}} U_R = e^{i\boldsymbol{k}(\alpha)\cdot\boldsymbol{a}} U_R^{-1} U_{\boldsymbol{a}} U_\alpha U_R = e^{i\boldsymbol{k}(\alpha)\cdot\boldsymbol{a}} U_{R^{-1}\boldsymbol{a}} U_\alpha$$
$$= U_\alpha U_{R^{-1}\boldsymbol{a}} = \exp[i\boldsymbol{k}(\alpha)\cdot R^{-1}\boldsymbol{a}] U_{R^{-1}\boldsymbol{a}} U_\alpha,$$

and thus

$$\boldsymbol{k}(\alpha)\cdot R^{-1}\boldsymbol{a} = R\boldsymbol{k}(\alpha)\cdot\boldsymbol{a} = \boldsymbol{k}(\alpha)\cdot\boldsymbol{a},$$

for all $\boldsymbol{a}$, R and α. But this implies $\boldsymbol{k}(\alpha)\equiv 0$, as desired.

If, on the other hand, the invariance group of π_A is $\tilde{\mathrm{E}}$ rather than E′, we may similarly prove that the operators U_g, $g\in\mathrm{E}$, commute not only with all gauge opearators U_α but also with all $U_\tau\equiv U_g$, $g=g(0,\tau,0,0,1)\in\mathrm{T}_1$. However, instead of $[U_\alpha, U_\tau]=0$ we can only prove

$$U_\alpha U_\tau = e^{ic\alpha\tau} U_\tau U_\alpha$$

with some constant c (see also Bargmann 1954), which means that the operators U_g, $g\in\tilde{\mathrm{E}}$, do not form a true representation of $\tilde{\mathrm{E}}$ unless $c=0$.

If configuration space is replaced by momentum space, we obtain from the models considered above other models, which are covariant under the homogeneous Galilei group H. The vectors φ_i in (25) are now defined by

$$\varphi_i(\boldsymbol{p}) = \begin{cases} C_i & \text{if } r_{i-1}\leqslant|\boldsymbol{p}|\leqslant r_i, \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

in place of (26), whereas equations (27) and (28) remain unchanged. These vectors are again rotation invariant, and since now boosts act as momentum space translations, $\varphi_i(\boldsymbol{p})\to\varphi_i(\boldsymbol{p}-m\boldsymbol{v})$, whereas space translations induce the transformations $\varphi_i(\boldsymbol{p})\to e^{-i\boldsymbol{p}\cdot\boldsymbol{a}}\varphi_i(\boldsymbol{p})$, it now follows that boosts are implemented whereas space translations are not. In this case, moreover, equation (23) implies as above that time translations are not implemented. The covariance group C of the models defined by (32) is thus $\mathrm{H}'\cong\mathrm{H}\otimes\mathrm{P}$, as for the models defined by (22). Again the operators U_g, $g\in\mathrm{H}'$, may be chosen to form a true representation of H′, as follows exactly as above for $\mathrm{C}=\mathrm{E}'$. (Note that E and H, and thus also E′ and H′, are isomorphic.)

That equations (22) and (32), although leading to the same covariance group H′, define unitarily inequivalent CAR representations π_A, is almost obvious, and could be proved formally with the help of the equivalence condition $A-B\in B(\mathscr{K})_2$ for quasi-free representations π_A and π_B. This remark applies also to the representations defined by equations (21) and (26), respectively.

Whether or not there are, besides the 'trivial' ones equivalent to either the Fock or the anti-Fock representation, also 'non-trivial' representations π_A which are covariant under the whole extended Galilei group G, is not known. The failure of our attempts to construct such representations might indicate that, as for the Poincaré group (Basarab-Horwath and Polley 1981), the answer to this question is negative†.

† *Note added in proof.* In the meantime, Basarab–Horwath (1982, private communication) has proved this.

3. Direct integrals of partially covariant representations

We shall now describe a method of constructing fully covariant CAR representations from partially covariant ones. The new representations are obtained as direct integrals, and are therefore always reducible. The method is formulated here more generally, for an arbitrary C^*-algebra $\mathfrak{A}$ and an arbitrary covariance group G, and is thus applicable also, e.g., to the CCR algebra of a Bose field, or to the relativistic case with the Poincaré group for G. A similar construction has been described by Basarab-Horwath *et al* (1979).

Let G be a topological group, K a subgroup and T a subset (not necessarily a subgroup) of G, such that every element $g \in \mathrm{G}$ may be decomposed in the form

$$g = kt \tag{33}$$

with unique $k \in \mathrm{K}$ and $t \in T$. Assume that k and t depend continuously on g. Keeping $g \in \mathrm{G}$ fixed and decomposing the group elements tg, with arbitrary $t \in T$, in the form

$$tg = k't' \tag{34}$$

analogous to (33), we obtain two mappings $\alpha_g : T \to T$ and $\beta_g : T \to \mathrm{K}$, defined by

$$\alpha_g(t) = t' \qquad \text{and} \qquad \beta_g(t) = k',$$

respectively. Then both t' and k' depend continuously on t and g, and for $g = e$ (the unit element of G) we get, in particular,

$$\alpha_e(t) = t, \qquad \beta_e(t) = e. \tag{35}$$

A measure $\mathrm{d}\mu(t)$ on T is called invariant, if for an arbitrary integrable function $f(t)$ on T the function $f(\alpha_g(t))$ is also integrable, and satisfies

$$\int_T f(\alpha_g(t))\,\mathrm{d}\mu(t) = \int_T f(t)\,\mathrm{d}\mu(t) \tag{36}$$

for arbitrary $g \in \mathrm{G}$. We assume that there exists such an invariant measure on T.

Now consider a C^*-algebra $\mathfrak{A}$ with a continuous representation τ_g, $g \in \mathrm{G}$, of the group G by automorphisms τ_g of $\mathfrak{A}$. Let π be a K-covariant representation of $\mathfrak{A}$—irreducible or not—on a Hilbert space $\mathscr{H}$, with implementing operators U_k, $k \in \mathrm{K}$, forming a true (rather than a projective) representation of K on $\mathscr{H}$, i.e.

$$U_k \pi(Y) U_k^* = \pi(\tau_k(Y)), \qquad U_{k_1} U_{k_2} = U_{k_1 k_2} \qquad \text{for all } k, k_1 \text{ and } k_2 \in \mathrm{K} \text{ and } Y \in \mathfrak{A}.$$

To construct a new, fully G-covariant representation $\hat{\pi}$ of $\mathfrak{A}$, we take the representation space $\hat{\mathscr{H}}$ to be a direct integral over T,

$$\hat{\mathscr{H}} = \int_{\oplus T} \mathscr{H}(t)\,\mathrm{d}\mu(t),$$

with $\mathscr{H}(t) = \mathscr{H}$ for all t, and an invariant measure $\mathrm{d}\mu(t)$. Vectors $\varphi \in \hat{\mathscr{H}}$ are thus represented by functions $\varphi(t)$ on T with values in $\mathscr{H}$, with the inner product being given by

$$(\varphi, \psi) = \int_T (\varphi(t), \psi(t))\,\mathrm{d}\mu(t).$$

The unitary representation $\hat{U}_g$, $g \in \mathrm{G}$, of G and the corresponding G-covariant representation $\hat{\pi}$ of $\mathfrak{A}$ on $\hat{\mathscr{H}}$ are then defined by

$$(\hat{U}_g\varphi)(t) = U_{\beta_g(t)}\varphi(\alpha_g(t)) = U_{k'}\varphi(t') \tag{37}$$

(with k' and t' from (35)) and

$$(\hat{\pi}(Y)\varphi)(t) = \pi(\tau_t(Y))\varphi(t), \tag{38}$$

respectively.

The operators $\hat{U}_g$ are isometric since, by (36) and (37),

$$\|\hat{U}_g\varphi\|^2 = \int_T \|(\hat{U}_g\varphi)(t)\|^2 \,\mathrm{d}\mu(t) = \int_T \|\varphi(\alpha_g(t))\|^2 \,\mathrm{d}\mu(t)$$

$$= \int_T \|\varphi(t)\|^2 \,\mathrm{d}\mu(t) = \|\varphi\|^2.$$

From (37) we also get, for arbitrary $g_1, g_2 \in \mathrm{G}$,

$$(\hat{U}_{g_1}\hat{U}_{g_2}\varphi)(t) = U_{k_1}(\hat{U}_{g_2}\varphi)(t_1) = U_{k_1}U_{k_2}\varphi(t_2) = U_{k_1k_2}\varphi(t_2)$$

with

$$tg_1 = k_1t_1, \qquad t_1g_2 = k_2t_2, \tag{39}$$

and

$$(\hat{U}_{g_1g_2}\varphi)(t) = U_{k_3}\varphi(t_3)$$

with

$$tg_1g_2 = k_3t_3. \tag{40}$$

By (39), on the other hand, we have

$$tg_1g_2 = k_1t_1g_2 = k_1k_2t_2.$$

Comparing this with (40), we find

$$t_2 = t_3, \qquad k_1k_2 = k_3$$

(due to the uniqueness of the decomposition (33)), and therefore

$$\hat{U}_{g_1}\hat{U}_{g_2} = \hat{U}_{g_1g_2}.$$

Together with $\hat{U}_e = 1$—which follows from (35) and (37)—the relations just proved imply that $\hat{U}_g$, $g \in \mathrm{G}$, is a unitary representation of G. The continuity of this representation will not be proved here, since it should be sufficiently plausible already from the assumed continuity properties of $\alpha_g(t)$ and $\beta_g(t)$ and equations (35).

The representation properties

$$\hat{\pi}(\alpha_1Y_1 + \alpha_2Y_2) = \alpha_1\hat{\pi}(Y_1) + \alpha_2\hat{\pi}(Y_2), \qquad \hat{\pi}(Y_1Y_2) = \hat{\pi}(Y_1)\hat{\pi}(Y_2),$$

$$\hat{\pi}(Y^*) = (\hat{\pi}(Y))^*, \qquad \hat{\pi}(1) = 1$$

of $\hat{\pi}$ follow from (38) by straightforward calculation. It is also obvious from (38) that $\hat{\pi}$ is always a reducible representation.

G-covariance of $\hat{\pi}$ under $\hat{U}_g$, as expressed by

$$\hat{U}_g\hat{\pi}(Y) = \hat{\pi}(\tau_g(Y))\hat{U}_g \tag{41}$$

for $Y \in \mathfrak{A}$, $g \in \mathrm{G}$, is verified as follows. By (37) and (38),

$$(\hat{U}_g \hat{\pi}(Y)\varphi)(t) = U_{k'}(\hat{\pi}(Y)\varphi)(t') = U_{k'}\pi(\tau_{t'}(Y))\varphi(t') = \pi(\tau_{k'}(\tau_{t'}(Y)))U_{k'}\varphi(t')$$

and, on the other hand,

$$(\hat{\pi}(\tau_g(Y))\hat{U}_g\varphi)(t) = \pi(\tau_t(\tau_g(Y)))(\hat{U}_g\varphi)(t) = \pi(\tau_t(\tau_g(Y)))U_{k'}\varphi(t'),$$

with $tg = k't'$ in both cases. The last relation implies

$$\tau_t(\tau_g(Y)) = \tau_{tg}(Y) = \tau_{k't'}(Y) = \tau_{k'}(\tau_{t'}(Y)),$$

so that, indeed, (41) follows.

In order to apply this construction to a specific example, with given group G and covariance subgroup K, one has to find the subset T of G 'complementary' to K and the corresponding decomposition (33) of group elements, to determine explicitly the mappings α_g and β_g defined via (34), and to look for an invariant measure $\mathrm{d}\mu(t)$ on T. We sketch this procedure here for the case of the extended Galilei group G, with K being one of the covariance groups $\tilde{\mathrm{E}}$, E′ or H′ of the models discussed in § 2. All calculations—using the multiplication law (10)—are straightforward, and are thus omitted here.

In the case $\mathrm{K} = \tilde{\mathrm{E}} \cong \mathrm{E} \otimes \mathrm{T}_1 \otimes \mathrm{P} = \{g(\alpha, \tau, \boldsymbol{a}, 0, R)\}$, we take $T = \mathrm{B} = \{g(0, 0, 0, \boldsymbol{v}, 1)\}$, the subgroup of pure boosts. Then a general group element,

$$g = g(\alpha, \tau, \boldsymbol{a}, \boldsymbol{v}, R) \in \mathrm{G}, \tag{42}$$

may be uniquely decomposed as $g = kt$, with

$$k = g(\alpha, \tau, \boldsymbol{a}, 0, R) \in \tilde{\mathrm{E}}, \qquad t = g(0, 0, 0, R^{-1}\boldsymbol{v}, 1) \in \mathrm{B}. \tag{43}$$

With g as in (42), the mappings $\alpha_g: \mathrm{B} \to \mathrm{B}$ and $\beta_g: \mathrm{B} \to \tilde{\mathrm{E}}$ act as follows:

$$\alpha_g: g(0, 0, 0, \boldsymbol{w}, 1) \to g(0, 0, 0, R^{-1}(\boldsymbol{w} + \boldsymbol{v}), 1), \tag{44}$$

$$\beta_g: g(0, 0, 0, \boldsymbol{w}, 1) \to g(\alpha + \tfrac{1}{2}mw^2\tau + m\boldsymbol{w} \cdot \boldsymbol{a}, \tau, \boldsymbol{a} + \tau\boldsymbol{w}, 0, R). \tag{45}$$

According to (44), an invariant measure on B is given by

$$\mathrm{d}\mu(g(0, 0, 0, \boldsymbol{w}, 1)) = \mathrm{d}^3\boldsymbol{w}. \tag{46}$$

For $\mathrm{K} = \mathrm{E}' \cong \mathrm{E} \otimes \mathrm{P} = \{g(\alpha, 0, \boldsymbol{a}, 0, R)\}$, we take $T = \{g(0, \tau, 0, \boldsymbol{v}, 1)\}$, the set of boosts combined with time translations, which is not a subgroup of G. In this case, the equations analogous to (43)–(46) read

$$k = g(\alpha, 0, \boldsymbol{a}, 0, R), \qquad t = g(0, \tau, 0, R^{-1}\boldsymbol{v}, 1),$$

$$\alpha_g: g(0, \sigma, 0, \boldsymbol{w}, 1) \to g(0, \sigma + \tau, 0, R^{-1}(\boldsymbol{w} + \boldsymbol{v}), 1),$$

$$\beta_g: g(0, \sigma, 0, \boldsymbol{w}, 1) \to g(\alpha + \tfrac{1}{2}mw^2\tau + m\boldsymbol{w} \cdot \boldsymbol{a}, 0, \boldsymbol{a} + \tau\boldsymbol{w}, 0, R),$$

$$\mathrm{d}\mu(g(0, \sigma, 0, \boldsymbol{w}, 1)) = \mathrm{d}^3\boldsymbol{w}\, \mathrm{d}\sigma,$$

respectively.

Finally, in the case $\mathrm{K} = \mathrm{H}' \cong \mathrm{H} \otimes \mathrm{P} = \{g(\alpha, 0, 0, \boldsymbol{v}, R)\}$, we set $T = \mathrm{T}_4 = \{g(0, \tau, \boldsymbol{a}, 0, 1)\}$, the subgroup of space–time translations. Then the equations corresponding to (43)–(45) become

$$k = g(\alpha + \tfrac{1}{2}mv^2\tau - m\boldsymbol{v} \cdot \boldsymbol{a}, 0, 0, \boldsymbol{v}, R), \qquad t = g(0, \tau, R^{-1}(\boldsymbol{a} - \tau\boldsymbol{v}), 0, 1),$$

$$\alpha_g: g(0, \sigma, \boldsymbol{b}, 0, 1) \to g(0, \sigma + \tau, R^{-1}(\boldsymbol{b} + \boldsymbol{a} - (\sigma + \tau)\boldsymbol{v}), 0, 1),$$

$$\beta_g: g(0, \sigma, \boldsymbol{b}, 0, 1) \to g(\alpha + \tfrac{1}{2}mv^2(\sigma+\tau) - m\boldsymbol{v}\cdot(\boldsymbol{b}+\boldsymbol{a}), 0, 0, \boldsymbol{v}, \boldsymbol{R}),$$

such that

$$\mathrm{d}\mu(g(0, \sigma, \boldsymbol{b}, 0, 1)) = \mathrm{d}^3\boldsymbol{b}\, \mathrm{d}\tau$$

is an invariant measure on T_4.

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